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1981 J. Phys. A: Math. Gen. 14 2251

(http://iopscience.iop.org/0305-4470/14/9/020)

J. Phys. A: Math. Gen. **14** (1981) 2251–2258. Printed in Great Britain

The redshift of hydrogen lines in a strong magnetic field

S H Patil
Department of Physics, Indian Institute of Technology, Bombay 400 076, India

Received 5 February 1981, in final form 23 March 1981

Abstract. We have analysed the low-lying energy levels of the hydrogen atom in a strong magnetic field. Isolation of the nearest singularity allows us to obtain expressions for energy levels in the strong-field limit. The resulting spectrum can simulate a redshifted hydrogenic spectrum and may have significant implications for astrophysical observations.

1. Introduction

The effect of a strong magnetic field on the energy levels of the hydrogen atom has acquired considerable importance in view of the possible existence of immensely large magnetic fields in astrophysical phenomena (Cohen *et al* 1970, Kadomtsev 1970, Mueller *et al* 1971). As such, the quadratic Zeeman shift in the hydrogen atom has been analysed in great detail, for the ground state and for some of the excited states (Avron *et al* 1977, 1979, Cabib *et al* 1972, Smith *et al* 1972, Brandi 1975, Galindo and Pascual 1976, Garstang 1977, Kanavi and Patil 1980). However, the general structure of the hydrogenic spectrum in a strong magnetic field is not yet well understood.

It is known (Cohen *et al* 1970, Kadomtsev 1970, Mueller *et al* 1971) that in the strong-field limit, the energy levels of the hydrogen atom are infinitely degenerate with respect to the z component of the angular momentum having values $0, -1, -2, \ldots$. In particular, it was shown by Hasegawa and Howard (1961) that the low-lying energy eigenvalues are, apart from the contribution of $\frac{1}{2}\gamma^{1/2}$ due to simple harmonic motion,

$$\varepsilon_{0,m} \xrightarrow[\gamma\to\infty]{} -\tfrac{1}{8}(\ln\gamma)^2 \tag{1}$$

for the ground state and

$$\varepsilon_{n,m} \xrightarrow[\gamma\to\infty]{} -\tfrac{1}{2}n^{-2} \tag{2}$$

for the excited states, with m denoting the degeneracy corresponding to the magnetic quantum number. They also showed that the excited states have an additional degeneracy of two corresponding to odd and even states for the motion in the z direction, and calculated the correction to the energies of the even states. They did not consider the odd states in any detail. The results stated above refer to the Hamiltonian

$$H = \tfrac{1}{2}p^2 - r^{-1} + \tfrac{1}{8}\gamma(x^2+y^2) + \tfrac{1}{2}\gamma^{1/2}L_z \tag{3}$$

where atomic units ($m = e = \hbar = 1$) have been used and $\gamma = c^{-2}H^2$ and $\gamma = 1$ corresponds to a field strength of 2.35×10^9 G.

Here, we calculate the corrections to the binding energies $\varepsilon_{n,m}$ of the low-lying states, for a large but finite magnetic field. This is done by analysing the analyticity properties of $\varepsilon_{n,m}$ as functions of the variable

$$\beta = \tfrac{1}{2}\gamma^{-1/2}. \tag{4}$$

We isolate the leading singular behaviour at $\beta = 0$ which allows us to calculate the leading finite-field correction to expression (2). It also provides insight into the structure of the energy levels. We find that the correction removes the degeneracy of the odd and even states and also the degeneracy of the magnetic quantum number for the odd states. The correction has n^{-3} dependence and can give rise to a redshifted Bohr spectrum. This raises an interesting possibility that some of the astrophysical redshifts may be due to the presence of strong magnetic fields.

2. Analyticity properties

For isolating the singular part of the energies, the analytic properties of the energy eigenvalues and the potential are crucially important. We first discuss the analytic properties of the effective potential.

2.1. *The effective potential*

The low-lying energy levels of the hydrogen atom in the presence of a strong magnetic field are given (Avron *et al* 1977, Hasegawa and Howard 1961, Patil 1980), apart from the simple harmonic contribution of $\frac{1}{2}\gamma^{1/2}$, by the one-dimensional motion in an effective potential

$$V_m(\beta, z) = -\frac{1}{\pi(4\beta)^{m+1}(m!)}\int \frac{\exp[-(x^2+y^2)/4\beta]}{(x^2+y^2+z^2)^{1/2}}(x^2+y^2)^m \, \mathrm{d}x\, \mathrm{d}y \tag{5}$$

where m is the magnitude of the magnetic quantum number and $\beta = \frac{1}{2}\gamma^{-1/2}$ as defined in (4). This expression is the Coulombic potential averaged over the simple harmonic motion in the xy plane. Using the Fourier representation

$$\frac{1}{r} = \lim_{\mu\to 0}\frac{1}{2\pi^2}\int \frac{\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{k^2+\mu^2}\,\mathrm{d}^3k \tag{6}$$

we obtain after carrying out the integrations

$$V_m(\beta, z) = \frac{2}{m!}(-a)^{m+1}\frac{\mathrm{d}^m}{\mathrm{d}a^m}\left(a^{-1/2}\int_0^\infty \mathrm{d}q\,\exp(-q^2 - 2q|z|a^{1/2})\right) \tag{7}$$

where $a = (4\beta)^{-1}$. This function is real analytic for real $\beta > 0$ and satisfies the Schwarz reflection principle:

$$V_m(\beta^*, z) = V_m^*(\beta, z). \tag{8}$$

2.2. *Energy eigenvalues for real $\beta > 0$*

It is observed that

$$V_m(\beta, z) \xrightarrow[\beta\to 0]{} -\frac{1}{|z|} \tag{9}$$

in which limit one has (Hasegawa and Howard 1961) energy eigenvalues (2) for even and odd eigenstates. For β positive, since

$$V_m(\beta, z) > -\frac{1}{|z|}, \tag{10}$$

the energy eigenvalues satisfy the inequality

$$\varepsilon_{n,m}(\beta) > -\tfrac{1}{2}n^{-2}. \tag{11}$$

Furthermore, it is clear from the nature of the Hamiltonian (3) that the particle remains bound for all positive γ, so that, by the Hellmann–Feynman theorem, one has

$$\frac{\partial \varepsilon_{n,m}(\beta)}{\partial \beta} = \langle \psi(\beta) | \frac{\partial V_m(\beta, z)}{\partial \beta} | \psi(\beta) \rangle \tag{12}$$

which exists for real $\beta > 0$. Thus, in this sense of the derivative existing, $\varepsilon_{n,m}(\beta)$ is real analytic for real $\beta > 0$.

2.3. *Im $E(\beta + i\varepsilon)$ for $\beta < 0$*

Here we derive an expression for Im $E(\beta + i\varepsilon)$ for $\beta < 0$. For this, we begin with the relation

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + V_m(\beta + i\varepsilon, z)\right)\psi(\beta + i\varepsilon, z) = E(\beta + i\varepsilon)\psi(\beta + i\varepsilon, z). \tag{13}$$

Multiplying the two sides by $\psi^*(\beta + i\varepsilon, z)$ and integrating by parts, one obtains

$$\mathrm{Im}\, E(\beta + i\varepsilon) = \int_{-\infty}^{\infty} dz\ \mathrm{Im}\ V_m(\beta + i\varepsilon, z)|\psi(\beta + i\varepsilon, z)|^2. \tag{14}$$

In obtaining this relation, surface terms have been thrown away. This however is reasonable in view of the fact that for the potential under consideration, $V_m(\beta, z) \to -1/|z|$ for $|z| \to \infty$, and hence one expects the wavefunction to vanish at infinity. It is certainly true for $\beta \to 0$.

We can easily deduce Im $V_m(\beta + i\varepsilon, z)$ from equation (7), for $\beta < 0$, with which equation (14) leads to

$$\mathrm{Im}\, E = \frac{\pi^{1/2} a^{m+1}}{m!}(-1)^m \frac{d^m}{da^m}\left(a^{-1/2}\int_{-\infty}^{\infty} dz |\psi(\beta + i\varepsilon, z)|^2 \exp(-az^2)\right) \qquad \text{for } \beta < 0 \tag{15}$$

where $a = (4\beta)^{-1}$. One could similarly obtain an expression for Im $E(\beta - i\varepsilon)$. It is observed, from equations (14) and (8), that

$$\mathrm{Im}\, E(\beta - i\varepsilon) = -\mathrm{Im}\, E(\beta + i\varepsilon) \tag{16}$$

which suggests that $\beta = 0$ is a branch point singularity of $\varepsilon_{n,m}(\beta)$.

2.4. *Dispersion relations*

The preceding analysis indicates that $\varepsilon_{n,m}(\beta)$ is analytic along the positive real axis but has a branch point at $\beta = 0$; one can take the branch cut along the negative real axis. Hence with the assumption that there are no other singularities on the first sheet away from the real axis, one can write the dispersion relations

$$\varepsilon_{n,m}(\beta)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im}\,\varepsilon_{n,m}(\beta'+\mathrm{i}\varepsilon)}{\beta'-\beta}\,\mathrm{d}\beta'. \tag{17}$$

Actually, our interest is primarily in separating out the leading singular behaviour at $\beta=0$, which is unaffected by the contributions from singularities away from the origin, as also by the possible need for subtractions.

3. Evaluation of $\varepsilon_{n,m}(\beta)$ for $\beta\to 0$

It is now possible to isolate the singular behaviour of $\varepsilon_{n,m}(\beta)$ at $\beta=0$ by using dispersion relations (17) with Im $\varepsilon_{n,m}(\beta+\mathrm{i}\varepsilon)$ obtained from equation (15). Indeed, since we are interested in the leading behaviour of Im $\varepsilon_{n,m}(\beta+\mathrm{i}\varepsilon)$ for $\beta\to 0$, we only need to know $\psi(\beta+\mathrm{i}\varepsilon, z)$ for $\beta\to 0$. These wavefunctions have been discussed by Haines and Roberts (1969), but require careful handling since they form a continuum at $\beta=0$. We consider the odd and even states separately.

3.1. $\varepsilon_{n,m}(\beta)$ for odd states

The states with the odd wavefunctions are relatively easy to handle since they form a discrete set at $\beta=0$. The un-normalised odd wavefunctions for $\beta=0$ are given by (Haines and Roberts 1969)

$$W_n(z)=\mathrm{e}^{-u/2}\sum_{r=0}^{\infty}\frac{(1-n)_r}{r!(r+1)!}u^{r+1}\qquad \text{for } z>0 \tag{18}$$

where

$$u=2z/n, \tag{19}$$

n takes values 1, 2, , . . . and

$$(l-n)_r=(1-n)(2-n)\ldots(r-n),\qquad (1-n)_0=1. \tag{20}$$

With these wavefunctions we evaluate Im $\varepsilon_{n,m}(\beta+\mathrm{i}\varepsilon)$ for $\beta\to 0_-$ from equation (15), and obtain

$$\operatorname{Im}\,\varepsilon_{n,m}(\beta+\mathrm{i}\varepsilon)\xrightarrow[\beta\to 0_-]{}\frac{4\pi(m+1)|\beta|}{n^3}. \tag{21}$$

Using this result in dispersion relations (17) and including expression (2) for the energy at $\beta=0$ (this is equivalent to using once-subtracted dispersion relations), we obtain

$$\varepsilon_{n,m}(\beta)\xrightarrow[\beta\to 0]{}-\tfrac{1}{2}n^{-2}(1+8n^{-1}(m+1)\beta\ln\beta) \tag{22}$$

where $\beta=\frac{1}{2}\gamma^{-1/2}$. The energy levels are seen to be raised for $\beta\neq 0$ in accordance with the general result (11).

3.2. $\varepsilon_{n,m}(\beta)$ for even states

The problem for the even states is greatly complicated by the fact that while for $\beta\neq 0$ one expects the states to be discrete—this is certainly true for $V(z)=-(|z|+\beta)^{-1}$, the even states form a continuum (Haines and Roberts 1969) for $\beta=0$. Hence it is not trivial to associate the discrete even states for $\beta\neq 0$ with the continuum even states for $\beta=0$.

To be more definite, let us designate the wavefunction by $\psi_n(\varepsilon_{n,m}(\beta),\beta,z)$. The interesting point made by Haines and Roberts (1969) is that the even eigenstates $\psi(\varepsilon,0,z)$ are admissible for all negative ε except for $\varepsilon=\varepsilon_{n,m}(0)$. We therefore prescribe that the correct limit of Im $\varepsilon_{n,m}(\beta+\mathrm{i}\varepsilon)$ for $\beta\to 0$ is obtained from equation (15) by the replacement

$$\psi_n(\varepsilon_{n,m}(\beta),\beta,z)\to\psi_n(\varepsilon_{n,m}(\beta),0,z)\qquad \text{for } \beta\to 0 \text{ but } \beta\neq 0. \tag{23}$$

This is plausible in view of the fact that this retains the correct asymptotic behaviour for $z\to\infty$. Also, it is found, *a posteriori*, that $(\varepsilon_{n,m}(\beta)-\varepsilon_{n,m}(0))$ is large compared with β. Furthermore, the correctness of this limit is verified explicitly for the potential $V(z)=-(|z|+\beta)^{-1}$ for which the wavefunctions are known (Haines and Roberts 1969). Now, one has (Haines and Roberts 1969) the un-normalised even solutions

$$W(\varepsilon_{n,m}(\beta),0,z)\xrightarrow[\beta\to 0]{}\mathrm{e}^{-u/2}\left(-\frac{1}{\alpha}+\frac{1}{\alpha-n}\sum_{r=0}^{\infty}\frac{(1-n)_r}{r!(r+1)!}u^{r+1}\right)\qquad \text{for } z>0 \tag{24}$$

where $\alpha=(-2\varepsilon_{n,m}(\beta))^{-1/2}$, and u and $(1-n)_r$ are defined in equations (19) and (20). Using these wavefunctions, we evaluate Im $\varepsilon_{n,m}(\beta)$ for $\beta\to 0_-$ from equation (15) and obtain

$$\operatorname{Im}\varepsilon_{n,m}(\beta)\xrightarrow[\beta\to 0_-]{}\frac{\pi}{2n^3}|\alpha-n|^2. \tag{25}$$

The dispersion relations (17) then imply

$$\varepsilon_{n,m}(\beta)\xrightarrow[\beta\to 0]{}\frac{1}{2n^3}\int_{-\infty}^{0}\frac{|\alpha(\beta')-n|^2}{\beta'-\beta}\,\mathrm{d}\beta' \tag{26}$$

where we have shown the β dependence of α explicitly. This is a self-consistency relation for $\beta\to 0$ and is satisfied for

$$\alpha(\beta)=n-2/\ln\beta\qquad \text{for } \beta\to 0 \tag{27}$$

which then leads to

$$\varepsilon_{n,m}(\beta)\xrightarrow[\beta\to 0]{}-\frac{1}{2n^2}\left(1+\frac{4}{n\ln\beta}\right) \tag{28}$$

where $\beta=\frac{1}{2}\gamma^{-1/2}$. As in the case of odd states, these levels are seen to be raised for $\beta\neq 0$. This expression is consistent with the quantum defect calculated by Hasegawa and Howard (1961).

The analysis discussed can also be carried out for the ground-state energy $\varepsilon_{0,m}$. In this case the wavefunction is

$$W_0(z)=\exp(-|z|/\alpha) \tag{29}$$

with which equation (15) gives

$$\operatorname{Im}\varepsilon_{0,m}(\beta)\xrightarrow[\beta\to 0_-]{}\pi/\alpha. \tag{30}$$

Using dispersion relations (17) as a consistency condition leads to

$$\varepsilon_{0,m}(\beta) \xrightarrow[\beta\to 0]{} -\tfrac{1}{2}(\ln \beta)^2 \tag{31}$$

where $\beta = \frac{1}{2}\gamma^{-1/2}$, in agreement with the known result (1).

4. Redshift of the spectrum

We have shown that the energy levels of the hydrogen atom in the presence of a strong magnetic field are given by, to leading orders,

$$E_{n,m}(\gamma) = \tfrac{1}{2}\gamma^{1/2} - \tfrac{1}{2}n^{-2}(1 + 8n^{-1}(m+1)\beta \ln \beta) \qquad n = 1, 2, \ldots \quad \text{for odd states,} \tag{32}$$

$$E_{n,m}(\gamma) = \tfrac{1}{2}\gamma^{1/2} - \frac{1}{2n^2}\left(1 + \frac{4}{n \ln \beta}\right) \qquad n = 1, 2, \ldots \quad \text{for even states} \tag{33}$$

$$E_{0,m}(\gamma) = \tfrac{1}{2}\gamma^{1/2} - \tfrac{1}{2}(\ln \beta)^2 \qquad \text{for the ground state} \tag{34}$$

where $\beta = \frac{1}{2}\gamma^{-1/2}$, and the simple harmonic energy has been included. The singular correction raises the energy levels of states with $n = 1, 2, \ldots$ which are non-degenerate for $\beta \neq 0$.

It may be observed that since the shifts decrease faster than n^{-2}, one set of the lines will be redshifted while the nature of the shifts of the other set will depend on the relative strengths of the shifts of the odd and even states. In view of this, we raise an interesting question: can some of the astrophysical redshifts be due to the presence of strong magnetic fields in stellar or galactic systems? This might be pertinent for the puzzle raised by the large redshifts in quasars.

For a quantitative discussion of the energy levels for strong but finite fields, we note that the energy levels of the odd states are no longer degenerate with respect to the magnetic quantum number. However, if we assume that the transition probability is maximum for some value $m = \bar{m}$, we may replace m by $\bar{m}$ for the purpose of describing the spectral lines.

To be specific, let us consider the Balmer-like lines. The wavelengths of the lines predicted by equations (32) and (33), as discussed above, may be given by

$$\lambda_{n\to 2} = \frac{911.76}{\frac{1}{4} - \frac{1}{8}b - (n^{-2} - cn^{-3})}\ \text{Å} \tag{35}$$

where $b = 8(\bar{m}+1)\beta|\ln \beta|$ and $c = 4(|\ln \beta|)^{-1}$. We apply this form to the hydrogenic Balmer lines observed for the quasar 3C273. The constants a and b are determined to be

$$b = 2.573 \times 10^{-1} \tag{36}$$

$$c = 4.197 \times 10^{-1} \tag{37}$$

so as to give the observed H_β wavelength of 5632 Å and H_γ wavelength of 5032 Å. With these values of b and c, we predict

$$\lambda_{6\to 2} = 4749\ \text{Å} \tag{38}$$

$$\lambda_{7\to 2} = 4590\ \text{Å} \tag{39}$$

for H_δ and H_ε lines which are in excellent agreement with the observed wavelengths of 4753 Å and 4595 Å respectively. The agreement is comparable to that of the Doppler shift predictions which, however, are a one-parameter description.

5. Discussion

We end our considerations with some comments.

(i) Our analysis of separating the singular part of the energy levels can also be used for potentials of the type

$$V(z) = -(|z|^N + \beta)^{-1/N}. \tag{40}$$

In particular, for $N = 1$, one obtains

$$\varepsilon_n = -\frac{1}{2n^2}\left(1 - \frac{4\beta}{n} - \frac{8\beta^2}{n}\ln\beta\right) \qquad n = 1, 2, \ldots \quad \text{for odd states} \tag{41}$$

$$\varepsilon_n = -\frac{1}{2n^2}\left(1 + \frac{2}{n\ln\beta}\right) \qquad \text{for } n = 1, 2, \ldots \quad \text{for even states} \tag{42}$$

$$\varepsilon_0 = -2(\ln\beta)^2 \qquad \text{for the ground state.} \tag{43}$$

These results agree with those of Haines and Roberts (1969), but in addition contain a higher-order correction for the odd states. This gives us confidence in the essential correctness of our approach. A similar analysis has been carried out for the three-dimensional truncated Coulomb potential (Mehta and Patil 1978). It is worth pointing out that in the problems we have considered, we have exploited the fact that Im $V(\beta, z)$ for $\beta < 0$ is easier to handle than the Re $V(\beta, z)$ which simplifies the analysis.

(ii) In the last section we described only the transitions to the odd $n = 2$ state. In general one has transitions to the even $n = 2$ state as well which may ultimately result in doublets of Balmer lines depending on the strengths of the transitions.

(iii) If the magnetic field varies significantly from one part of the stellar or galactic body to another, the observed lines will acquire a linewidth. The spectral lines of quasars do indeed have linewidths of the order of 50 Å. Variation in the direction of the magnetic field can also diffuse the polarisation of the emitted radiation.

(iv) Our analysis has considered only the hydrogenic spectrum. Even if the quasar redshifts are not due to strong magnetic fields, it would be interesting to establish the existence of strong magnetic fields in astrophysical phenomena by identifying the redshifted hydrogen lines.

(v) It should be noted that for large n, the Coulombic corrections in equations (32) and (33) become small. Hence our results may be applicable to atomic states with large n, even for moderate values of the magnetic field.

Acknowledgment

The author acknowledges helpful discussions with Dr C H Mehta.

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